International Journal of Theoretical Physics, Vol. 33, No. 2, 1994

Bosonic String in the Hot Schwarzschild Geometry

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Received May 14, 1993

We study the bosonic string in the Schwarzschild–de Sitter black hole, which has a black hole horizon as well as a cosmological horizon. This generalizes the bosonic string in the cold Schwarzschild black hole already studied.

1. INTRODUCTION

In recent years there has been interest in studying strings in the context of quantum gravity. A first step in this regard was the investigation of string quantization in the Rindler spacetime and the Hawking–Unruh effect in string theory (de Vega and Sanchez, 1987; Sanchez, 1987). Sanchez and de Vega (1987) developed a general method of string quantization in curved spacetime taking into account the strong curvature effect of the geometry. They treat the spacetime metric exactly and the string excitations small as compared with the energy scale of the geometry. They applied their method to the de Sitter and Schwarzschild black hole geometries (de Vega and Sanchez, 1988*a*). We would like to apply their method to the Schwarzschild–de Sitter black hole (Gibbons and Hawking, 1977). We call the Schwarzschild–de Sitter black hole a hot Schwarzschild black hole, since the de Sitter spacetime has been interpreted as being hot (Gasperini, 1988).

2. BOSONIC STRING IN THE HOT SCHWARZSCHILD GEOMETRY

The hot Schwarzschild line element is given by

$$ds^2 = -\Sigma\, dX^{02} + \Sigma^{-1}\, dR^2 + R^2\, d\Omega^2 \tag{1}$$

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where

$$d\Omega^2 = d\theta^2 + \sin^2\theta \, d\phi^2$$

$$\Sigma = 1 - 2M/R - R^2/a^2, \qquad a = (3/\Lambda)^{1/2}$$

The parameters M and Λ represent the mass of the source and the cosmological constant, respectively. If $\Lambda < (9M^2)^{-1}$, $\Sigma = 0$ gives two positive values. The smaller value gives the black hole horizon while the larger one is similar to the cosmological horizon in the de Sitter space.

The D-dimensional generalization of the hot Schwarzschild metric is given by

$$ds^2 = -[1 - (1 - \Sigma)^{D-3}] \, dX^{02} + \frac{dR^2}{1 - (1 - \Sigma)^{D-3}} + R^2 \, d\Omega^2_{D-2} \tag{2}$$

where R is the radial coordinate and $d\Omega^2_D$ is the line element of the D-dimensional unit sphere.

The string equations of motion in the hot Schwarzschild metric are given by

$$\begin{aligned} \partial^2 X^0 + \frac{K}{\beta(R)} (1 - \Sigma)^{D-2} \, \partial_\mu X^0 \, \partial^\mu R &= 0 \\ \partial^2 R - \frac{K}{\beta(R)} (1 - \Sigma)^{D-2} (\partial_\mu R)^2 - \beta[R(\partial\mu \, \Omega^i)^2 - K(1 - \Sigma)^{D-2} (\partial_\mu X^0)^2] &= 0 \\ \partial^2 \Omega^i + \frac{2}{R} \partial^\mu R \, \partial_\mu \Omega^i - \Omega^i (\Omega^j \, \partial^2 \Omega^j) &= 0 \end{aligned} \tag{3}$$

where

$$\begin{aligned} (\Omega^i)^2 &= 1, \qquad 1 \lesssim i \lesssim D - 2 \\ \beta(R) &= 1 - (1 - \Sigma)^{D-3} \\ K &= \frac{D-3}{2R(1 - \Sigma)} \end{aligned} \tag{4}$$

The constraint equations are

$$T_{\pm\pm} = \frac{1}{\beta} (\partial \pm R)^2 - \beta(\partial \pm X^0)^2 + R^2 (\partial \pm \Omega^i)^2 = 0 \tag{5}$$

Following the general method developed by Sanchez and de Vega (1987), we set

$$X^A(\sigma, \tau) = g^A(\tau) + \eta^A(\sigma, \tau) + \xi^A(\sigma, \tau) + \cdots \tag{6}$$

where g^A is an exact solution of the equation of motion of the string. g^A, η^A obey the linearized perturbation around g^A, and ξ^A satisfies the second-order perturbation around g^A. Since g^A obeys the center-of-mass equation of the string, we have

$$m^2 + \frac{1}{\alpha'^2\beta}(\dot{g}^R)^2 - \frac{E^2}{\beta^2} + \frac{L^2}{g_R^2} = 0 \tag{7}$$

where m is the mass, α' is the tension, E is the energy, and L is the angular momentum of the string. Equation (7) describes the motion in the plane defined by the spherical coordinates $\theta^i = \pi/2$. That is, we have

$$g^1 = \cos\phi, \qquad g^2 = \sin\phi, \qquad g^a = 0, \qquad a > 2, \qquad \dot{\phi} = \frac{\alpha' L}{g_R^2} \tag{8}$$

Equations (7)–(8) are solvable by

$$\tau = \int g_R \frac{dS}{[E^2 - \beta(S)(m^2 + L^2/S^2)]^{1/2}} \tag{9}$$

We recognize here an effective potential

$$V_{\text{eff}}(g) = \left(m_2 + \frac{L^2}{g^2}\right)\left[1 - \left(\frac{R(1-\Sigma)}{g}\right)^{D-3}\right] \tag{10}$$

The absorption or elastic scattering of the particle by the hot Schwarzschild black hole will depend on the initial energy and momentum.

The first-order quantum fluctuations $\eta^A(\sigma, \tau)$ satisfy the following equations:

$$\partial^2\eta^* + \left[\frac{\beta''}{2\beta}(\dot{g}^{R^2} + \alpha'^2E^2) - \frac{L^2\alpha'^2}{(g^R)^4}\right]\beta\eta^* + \frac{\beta'}{\beta}(\dot{g}^R\dot{\eta}^* + \alpha' E\dot{\eta}^0 - 2g^R\dot{g}^i\dot{\eta}^i) = 0$$
$$\partial^2\eta^i - \frac{2\dot{g}^R}{g^R}\dot{\eta}^i - 2\dot{g}^i\,\partial_\tau\left(\frac{\beta\eta^*}{g^R}\right) - \frac{L^2\alpha'^2\eta^i}{(g^R)^4} - 2g^i\dot{g}^j\dot{\eta}^j\delta = 0 \tag{11}$$

where the R^* coordinate is defined by

$$R^* = R + [R(1-\Sigma)]^{D-3}\int^R \frac{dR'}{R'^{D-3} - [R(1-\Sigma)]^{D-3}} \tag{12}$$

Equations (11) can be solved in the asymptotic region $\tau \to \pm\infty$. At $\tau \to \pm\infty$, the center of mass is very far from the center of forces and the spacetime is practically flat. We set

$$\eta^A(\sigma, \tau) = \frac{\beta^A(\sigma, \tau)}{g^R(\tau)} \tag{13}$$

and expand

$$\alpha^A(\sigma, \tau) = \sqrt{\alpha'} \sum_{n \in z} \sum_{B=0}^{D-1} f_{n,\pm}^{AB}(\tau)(\alpha_{n,\pm}^B e^{-in\sigma} + \tilde{\alpha}_{n,\pm}^B e^{in\sigma}) \tag{14}$$

where the function $f_{n,\pm}^{A,B}(\tau)$ satisfies the fluctuation equations (11) with the boundary conditions

$$\lim_{\tau \to \pm\infty} \int_{n,\pm}^{A,B}(\tau) = e^{-in\tau}\delta^{AB} \tag{15}$$

The choice of a positive frequency factor $e^{-in\tau}$ in equation (15) corresponds to the "in" particle for $\tau \to -\infty$ or the "out" particle state for $\tau \to +\infty$. Since we have free oscillators for both $\tau \to -\infty$ and $\tau \to \infty$, we have

$$\lim_{\tau \to \mp\infty} \int_{n,\pm}^{A,B}(\tau) = A_{n,\pm}^{A,B} e^{-in\tau} + B_{n,\pm}^{A,B} e^{in\tau} \tag{16}$$

where the coefficients $A_{n,\pm}^{A,B}$ and $B_{n,\pm}^{A,B}$ depend on the detailed form of equations (1) for all τ. From equations (14)–(16), it is evident that the outgoing and ingoing operator modes are related by the Bogolubov transformation.

$$\begin{aligned} \beta_{n,+}^A &= \sum_B (A_{n,+}^{A,B} \beta_{n,-}^B + B_{n,+}^{A,B} \beta_{-n,-}) \\ \tilde{\beta}_{n,+}^A &= \sum_B (A_{n,+}^{A,B} \tilde{\beta}_{n,-}^B + B_{n,+}^{A,B} \beta_{-n,-}^B) \end{aligned} \tag{17}$$

We see two main effects in the transition between the string ingoing and outgoing modes produced by the hot Schwarzschild black hole:

(i) Polarization changes in the modes.

(ii) Pair mode creation. Each pair is formed by modes of opposite chirality.

We find (de Vega and Sanchez, 1988*b*,*c*) for the modes orthogonal to the scattering plane

$$B_{n,+}^{ij} = \delta^{ij} B_{n,+}, \qquad 2 < i, j \le D - 1 \tag{18}$$

Therefore, we find for the pair creation amplitude

$$\langle \overleftarrow{n}_{\text{out}}, \overrightarrow{n}_{\text{out}} | 0_{in} \rangle = B_{n,+} \tag{19}$$

where $|\overleftarrow{n}_{\text{out}}, \overrightarrow{n}_{\text{out}}\rangle$ stands for an outgoing state with the left and the right nth modes occupied. For an excited initial state, we have

$$\overrightarrow{m}_{\text{out}}, \overleftarrow{n}_{\text{out}}, \overrightarrow{n}_{\text{out}} | \overrightarrow{e}_{\text{in}} = \delta m_e B_n, \qquad e \neq n$$

Following de Vega and Sanchez (1988*c*), we find the coefficients A_n and B_n for large impact parameters as

$$A_n = i\alpha'\left(\frac{R(1-\Sigma)}{b}\right)^{D-3}\frac{P\sqrt{\pi}}{b}\frac{\Gamma(D/2+1)}{\Gamma((D+1)/2)}\left(1+\frac{D-1}{D}\frac{m^2}{P^2}\right),\quad b \gg R(1-\Sigma)$$

$$\beta_n = P\alpha'\frac{[R(1-\Sigma)]^{D-3}}{b^{D-2}}F\left(\frac{m}{P},\frac{nb}{\alpha' P}\right) \tag{20}$$

$$\simeq P\alpha'\frac{[R(1-\Sigma)]^{D-3}}{b^{D-2}}\left(\frac{2nb}{\alpha' P}\right)^{D/2}\exp\left[-\left(\frac{2mb}{\alpha' P}\right)\right]$$

where the function $F(x, y)$ is defined as a double integral (de Vega and Sanchez, 1988*c*). It should be noted that B_n is appreciably different from zero when the characteristic interaction time is of the same order of magnitude as the vibration time of the mode. That is,

$$\frac{b}{\alpha' P} \simeq \frac{2\pi}{n} \quad \text{or} \quad n \simeq \frac{\alpha' P}{b} \tag{21}$$

When $n \gg \alpha' P/b$, B_n is very small. At zeroth order we find for the center-of-mass cross section

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{CM}} = \left(\frac{b}{\sin\theta}\right)^{D-3}\frac{db}{d\theta}$$

$$\underset{\theta\to 0}{=} \frac{[4GM\Gamma(D/2-1)\pi^{2-D/2}\{1+[(D-3)/(D-2)]m^2/P^2\}](D-2)/(D-3)}{(D-3)\theta^{D-1}+1/(D-3)} \tag{22}$$

where G is the gravitational constant and M is the mass of the hot black hole. This generalizes the Rutherford formula for $D = 4$. Taking into account the first quantum correction gives

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{elastic}} = \left(\frac{d\sigma}{d\Omega}\right)_{\text{CM}}\left(1-\sum_{n'\neq m} B_{n'}^2\right) \tag{23}$$

If the initial stage and the final state are of the same nth modes, then we get an elastic cross section. The inelastic cross section is proportional to $|B_n|^2$.

3. CONCLUSION

The results obtained in this paper correspond to the results obtained in the case of the Schwarzschild black hole when we set $\Lambda = 0$. Therefore we observe that the physical results remain the same whether we consider the string in the Schwarzschild black hole or in the Schwarzschild black hole in the de Sitter spacetime, which has recently attracted renewed interest as a model of the inflationary stage of the early universe.

ACKNOWLEDGMENT

I would like to thank Prof. Abdus Salam for the hospitality which we provided at ICTP, Trieste, Italy.

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